Math-601D-201: Lectures 23–24. Hörmander's solution of the $\overline{\partial}$-equation

Charles Favre

charles.favre@polytechnique.edu

April 2nd, 2020

Main theorem

Theorem

Let $\Omega \subset \mathbb{C}^n$ be any pseudo-convex domain.
For any smooth $(p, q+1)$-forms f on Ω satisfying $\bar{\partial} f = 0$, there exists a smooth (p, q)-form u such that $\bar{\partial} u = f$.

1. Solve $\bar{\partial} u = f$ in a suitable L^2-space using functional analysis techniques
2. Prove the solution is smooth when f is smooth (regularity of the $\bar{\partial}$-equation)

The set-up

$\varphi \in \mathcal{C}^0(\Omega)$. Consider the Hilbert space:

$$L^2_{p,q}(\varphi) = \left\{ \sum u_{IJ} dz_I \wedge d\bar{z}_J,\ |u|^2_\varphi := \int \sum_{I,J} |u_{IJ}|^2 e^{-\varphi} < \infty \right\}$$

$\mathcal{D}_{p,q}(\Omega)$ (smooth compactly supported (p,q)-forms) are dense in $L^2_{p,q}(\varphi)$

$$\boxed{L^2_{p,q}(\varphi_1) \xrightarrow[T]{\bar\partial} L^2_{p,q}(\varphi_2) \xrightarrow[S]{\bar\partial} L^2_{p,q}(\varphi_3)}$$

T and S are densely defined closed (unbounded) operators

A priori estimates

$T: H_1 \to H_2$ closed and densely defined.

Theorem

F a closed subspace of H_2 containing $Ran(T)$. The following are equivalent:

- *$Ran(T) = F$;*
- *there exists $C > 0$ such that*

$$\|f\|_{H_2} \leq C \, \|T^* f\|_{H_1} \text{ for all } f \in F \cap Dom(T^*)$$

Observation: $\ker(T)^\perp = \overline{Ran(T^*)} = Ran(T^*)$

Set-up (II)

We want to prove for all $f \in Dom(T^*) \cap Dom(S)$

$$\boxed{|f|^2_{\varphi_2} \leq C\left(|T^*f|^2_{\varphi_1} + |Sf|^2_{\varphi_3}\right)}$$

Theorem

The space $\mathcal{D}_{p,q}(\Omega)$ is dense in $Dom(T^*) \cap Dom(S)$ for the graph norm

$$f \mapsto |f|_{\varphi_2} + |T^*f|_{\varphi_1} + |Sf|_{\varphi_3}$$

for appropriate $\varphi_1, \varphi_2, \varphi_3$.

Constraints on $\varphi_1, \varphi_2, \varphi_3$

- Choose $\eta_j \in \mathcal{D}(\Omega)$, $0 \leq \eta_j \leq 1$ such that for any compact set K, $\eta_j|_K \equiv 1$ for all $j \gg 0$.
- Choose $\psi \in \mathcal{C}^\infty(\Omega)$ such that

$$\sum_{k=1}^{n} \left| \frac{\partial \eta_j}{\partial \bar{z}_k} \right|^2 \leq e^{\psi} \text{ for } j = 1, 2, \cdots$$

- Set $\varphi_1 = \varphi - 2\psi$, $\varphi_2 = \varphi - \psi$, $\varphi_3 = \varphi$

Freedom to choose any φ!!!

Some computations

If $f = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$

$$|\bar{\partial} f|^2 = \sum_{I,J} \sum_j \left| \frac{\partial f_{IJ}}{\partial \bar{z}_j} \right|^2 - \sum_{IK} \sum_{jl} \frac{\partial f_{I,jK}}{\partial \bar{z}_l} \times \overline{\frac{\partial f_{I,lK}}{\partial \bar{z}_j}}$$

$$(-1)^{p-1} e^{\psi} T^* f = \sum_{IK} \sum_j \delta_k (f_{I,jK}) dz_I \wedge d\bar{z}_K + \sum_{IK} \sum_j f_{I,jK} \frac{\partial \psi}{\partial z_j} dz_I \wedge d\bar{z}_K$$

with $\delta_j h = e^{\varphi} \frac{\partial (h e^{-\varphi})}{\partial z_j} = \frac{\partial h}{\partial z_j} - h \frac{\partial \varphi}{\partial z_j}$

The key estimate

If $f = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$

$$\sum_{IK} \int \sum_{jk} f_{I,jK} \bar{f}_{I,kK} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} + \sum_{IJ} \sum_{j=1}^{n} \int \left| \frac{\partial f_{IJ}}{\partial \bar{z}_j} \right|^2 e^{-\varphi}$$

$$\leq 2|T^* f|_{\varphi_1} + |Sf|^2_{\varphi_3} + 2 \int |f|^2 |\partial \psi|^2 e^{-\varphi}$$

Solution to the $\bar{\partial}$-equation in L^2

Theorem

$\Omega \subset \mathbb{C}^n$ pseudo-convex
For any $f \in L^2_{p,q+1}(\Omega, loc)$, $\bar{\partial} f = 0$, there exists
$u \in L^2_{p,q}(\Omega, loc)$ such that $\bar{\partial} u = f$.

Sobolev space: $W^s = \{f \in L^2, \partial^I f \in L^2, \forall |I| \leq s\}$

Theorem

Ω pseudoconvex
Let $f \in W^s_{p,q+1}(\Omega, loc)$ such that $\bar\partial f = 0$. If

- *$\bar\partial u = f$, and*
- *$u \in Ran(T^*)$*

then $u \in W^{s+1}_{p,q+1}(\Omega, loc)$.

$\longrightarrow$ conclude using Sobolev embedding $W^{s+2n}_{loc} \subset \mathcal{C}^s$.